# Parallel (1 + $\epsilon$ )-Approximate Multi-Commodity Mincost Flow in Almost Optimal Depth and Work

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### Minimum Cost Flow

- Input: a directed graph with edge capacities and costs, two nodes s, t
  - For this talk: capacities and costs are non-negative integers
- Output: Max flow with Min cost.
  - Max flow: maximum (fractional) number of s-t flow path that respects capacity
  - Min cost: minimum summation of cost (length) of all the flow path
- Special cases:
  - Max flow
  - SSSP
  - Reachability
  - ...

## Parallel Computing

- PRAM: shared memory parallel model
  - Work: total number of unit operations
  - Depth: longest dependency chain of the algorithm
- Work-efficient: Work =  $\tilde{O}$ (best sequential running time)
- Highly parallelizable: Depth=  $\tilde{O}(1)$  and work-efficient
  - For this talk: Depth=  $n^{o(1)}$  and work-efficient

What problems are highly parallelizable?

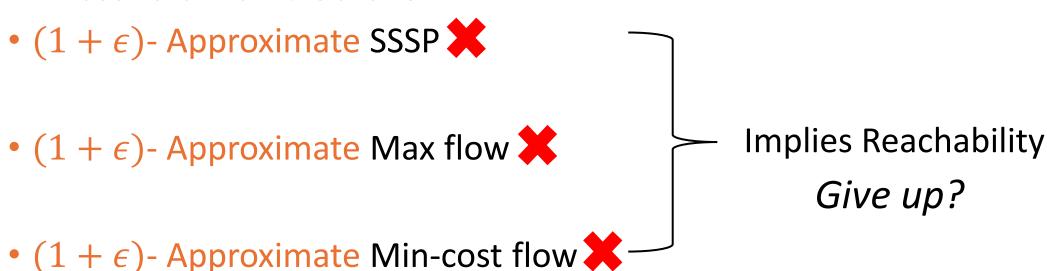
- Reachability?
  - Decades of attempts ... stuck at  $n^{0.5+o(1)}$  depth [JLS'19]
  - Lower bound of  $n^{1/4}$  [BH'23] for shortcut type algorithms
  - Seems far from the answer ...
- SSSP 💢

Max flow

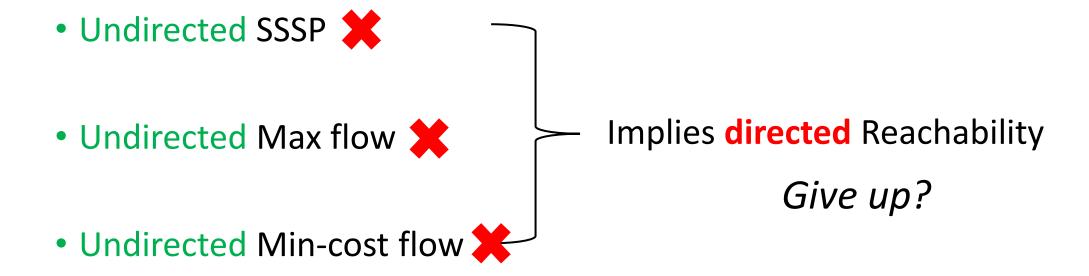
Give up?

Min-cost flow

- Reachability?
  - Decades of attempts ... stuck at  $n^{0.5+o(1)}$  depth [JLS'19]
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  - Seems far from the answer ...



• Undirected Reachability:  $\tilde{O}(1)$  depth!



- Undirected  $(1 + \epsilon)$ -Approximate SSSP
  - [Cohen, STOC'94]  $n^{o(1)}$  depth via hopset ("distance shortcut")
  - [Li, STOC'20]  $\tilde{O}(1)$  depth via oblivious routing
- Undirected  $(1 + \epsilon)$ -Approximate Max flow
  - [Sherman, FOCS'13] First  $\tilde{O}(m)$  work sequential (congestion approximator + MWU)
  - [Agarwal, Khanna, Li, Patil, Wang, White, Zhong, SODA'24]  $ilde{O}(1)$  depth following Sherman's approach
  - Sherman's approach does not work for Vertex Capacitary or Min-Cost
- Undirected  $(1 + \epsilon)$ -Approximate Min-cost flow (with vertex capacity & cost)
  - [Bernstein, Gutenberg, Saranurak, FOCS'21]  $First \ m^{1+o(1)} \ work \ sequential \ (decremental \ SSSP + MWU)$
  - [Chen, Kyng, Liu, Peng, Gutenberg, Sachdeva, FOCS'22]  $m^{1+o(1)}$  work (IPM for m iterations)
  - Our result:  $m^{1+o(1)}$  work,  $n^{o(1)}$  depth via "LC-flow shortcut"
  - \* Approximate means  $(1 \epsilon)$  flow with  $(1 + \epsilon)$  cost for maximum flow

#### Overview

- 1. Low-step  $(1+\epsilon)$ -Approximate min-cost flow in  $m^{1+o(1)}$  work and  $n^{o(1)}$  depth [Haeupler, Hershkowitz, Saranurak, STOC'23]
  - There exists an approximate flow and that every flow path has  $n^{o(1)}$  edges.
  - Main techniques: flow path LP with  $\exp(n^{o(1)})$  constraints + MWU in  $n^{o(1)}$  rounds
- 2. Add "LC-flow shortcut" edges H to the graph G so that
  - Every flow in G maps to a low-step flow in  $G \cup H$
  - Every flow in  $G \cup H$  maps to a flow in G
  - Flow mapping has
    - Cost slack  $(1 + \epsilon)$
    - Flow value (congestion) slack  $n^{o(1)}$

Our focus

- 3. We get: min-cost flow with  $(1+\epsilon)$ -cost approximation and  $n^{o(1)}$ -flow approximation
  - Use MWU to boost the flow approximation to  $(1-\epsilon)$

### LC-Flow shortcut: Plan

- 1. A "distance shortcut" preserving  $(1 + \epsilon)$  distance (no congestion) [Cohen, STOC'94]
  - Low diameter decomposition
- 2. A "congestion shortcut" preserving only  $n^{o(1)}$  congestion (no cost)
  - Expander decomposition hierarchy
- 3. A "LC-flow shortcut" preserving both  $n^{o(1)}$  congestion and  $(1 + \epsilon)$  distance
  - Combining the ideas of 1 and 2
  - Length-constraint expander decomposition hierarchy

### Distance Shortcut [Cohen, STOC'94]

- LDD: Low-diameter decomposition (with a parameter d):
  - Partition vertex set into clusters with diameter at most d
  - Any path (we care about) of length d crosses at most  $\tilde{O}(1)$  clusters
    - To be precise: each edges is crossing clusters with probability  $\tilde{O}(\frac{1}{d})$
- A simple distance shortcut with slack  $\tilde{O}(1)$ 
  - For every scale of d: compute LDD with parameter d, add a d-star to each cluster
- It is possible to boosting distance slack to  $(1 + \epsilon)$

### **Congestion Shortcut**

- Expander decomposition (with a parameter  $\phi = 1/n^{o(1)}$ ):
  - Partition vertex sets into clusters with expansion at least  $\phi$  (any degree-respecting demand can be routed with congestion  $1/\phi$  satisfying)
  - At most  $\tilde{O}(\phi \cdot m)$  crossing cluster edges
- Add a star to each cluster (capacity equals to degree): congestion slack  $n^{o(1)}$
- For crossing cluster edges: run terminal expander decomposition
  - Repeat  $O(\log_{\frac{1}{\phi}} n)$  layers

### Combining Distance and Congestion: LC-Flow Shortcut

- How to define a cluster? (Think of combining Low-diameter and Expander)
  - Every degree-respecting demand can be routed by a flow with congestion  $\phi$  and average cost d
- What is the guarantee of decomposition?
  - $\tilde{O}(m\phi)$  edges must cross clusters, and then
  - Each edge becomes crossing with probability  $\tilde{O}(\frac{1}{d})$
- Making this intuition to be continuous: "crossing" -> "increasing length"
  - Length-constraint expander decomposition [Haeupler, Raecke, Ghaffari STOC'22]
- Add stars with (1) degree as capacity (2) d as cost -> LC-Flow shortcut

### Summary and Open Problems

- Undirected  $(1 + \epsilon)$ -Approximate Min-cost flow (with vertex capacity & cost)
  - $m^{1+o(1)}$  work  $n^{o(1)}$  depth

#### Open problems:

- How to improve to  $\tilde{O}(m)$  work and  $\tilde{O}(1)$  depth? (Even sequential is unknow)
  - Shortcut based approach cannot work,  $(1+\epsilon)$ -hopset has lower bound  $n^{o(1)}$
  - [Li, STOC'20] circumvent the barrier for SSSP using oblivious routing
- Removing "undirected" or "approximation"?
  - Requires solving reachability (shortcut upper / lower bound)
  - Match the performance of reachability ( $\tilde{O}(m)$  work  $n^{0.5+o(1)}$  depth)
    - Linear size LC-flow shortcut for directed graph with depth  $n^{1-\epsilon}$ ?
    - Min-cost flow in almost linear work and h depth for h-length bound flow path?